10. SYSTEMS OF LINEAR EQUATIONS

§10.1. Systems of Simultaneous Equations

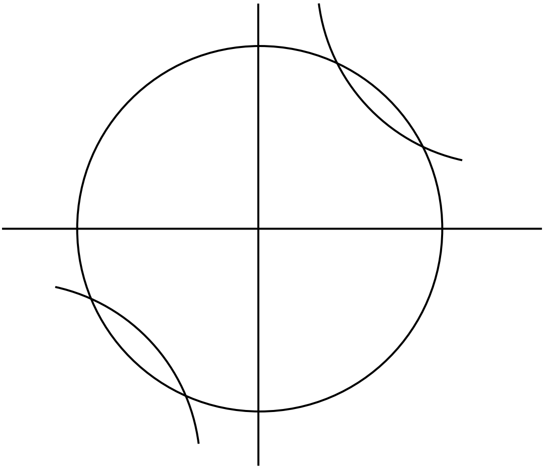
If we have two equations in two variables, x and y, we may be able to solve them **simultaneously**, that is, find one or more pairs of values which satisfy *both* equations at the same time. Geometrically an equation $f(x, y) = 0$ can be represented by a curve in the plane. A second equation $g(x, y) = 0$ will give another curve. Where these curves intersect, if indeed they do, will give points that correspond to pairs of x and y that comprise the solutions. In principle this can be extended to any number of equations and ant number of variables, and we call the set of such equations a **system** of equations. Here we will stick to two equations in two variables.

Example 1: Find the number of solutions to the system

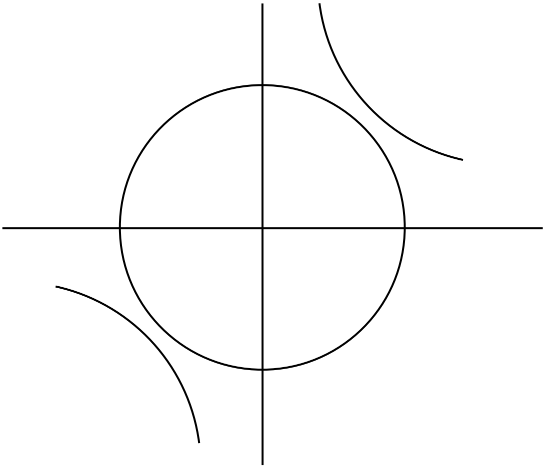
$$\left.\begin{aligned} x^2 + y^2 &= 13 \\ xy &= 6 \end{aligned}\right\} .$$

Solution: The equation $x^2 + y^2 = 13$ represents a circle with radius $\sqrt{13}$. The equation $xy = 6$ represents a hyperbola, $y = \frac{6}{x}$.

Drawing both on the same set of axes we get:



Now, if we’d merely sketched the curves we might not be sure whether the curves intersect at all. Perhaps the situation might be as follows.



If we'd plotted points we'd almost certainly have stumbled on the solutions:

$$x = 2, y = 3; x = -2, y = -3;$$
$$x = 3, y = 2; x = -3, y = -2.$$

However, accidentally finding solutions does not constitute *solving* the system.

In this case we can eliminate y, by writing $y = \frac{6}{x}$ from the second equation and substitute this into the first equation. We get $x^2 + \frac{36}{x^2} = 13$. This is now one equation in one variable. Eliminating variables so as to reduce the number of equations and variables is a good technique for solving systems of equations, but it's not always possible.

Multiplying by x^2 we get $x^4 + 36 = 13x^2$ and hence

$$x^4 - 13x^2 + 36 = 0.$$

This is a fourth degree polynomial equation and, in general, they're quite hard to solve. But this one happens to be a quadratic in x^2. If we put $X = x^2$ the equation becomes $X^2 - 13X + 36 = 0$.

Luckily this quadratic is very easy to solve since it factorises nicely:

$$(X - 4)(X - 9) = 0, \text{ so } X = 4 \text{ or } 9.$$

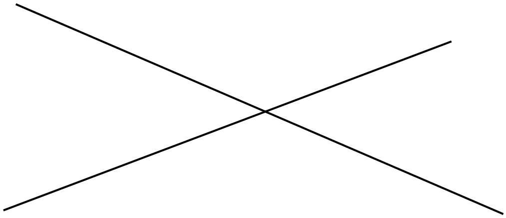
So $x^2 = 4$ or 9, giving $x = \pm 2$ or ± 3. The corresponding values of y are obtained by substituting into $y = \frac{6}{x}$.

§10.2. Systems of Linear Equations

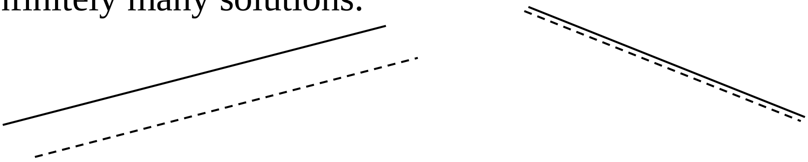
We'll now concentrate on the simplest type of system of equations – systems of two linear equations in two variables. A linear equation is one that represents a straight line, that is, an equation of the form:

$$ax + by + c = 0.$$

If a linear equation represents a straight line then two linear equations represent two straight lines, and typically they intersect in a single point, giving exactly one solution.



The exceptions are where the lines are parallel. If they're parallel, but different there's no solution. And if the equations represent the same straight line there are infinitely many solutions.



The technique of eliminating one variable to get one equation in one variable works well here. But often it's easier to do this by adding or subtracting a multiple of one equation to or from the other.

Example 2: Find numbers x, y such that

$$x + 2y = 20$$
$$5x - 2y = 40$$

Solution: If we add the two equations we get $6x = 60$, that is, $x = 10$.

What we have done is to combine the two equations in such a way that we have eliminated y. We can then solve for x.

We now substitute $x = 10$ into either of the original two equations to get $y = 5$. So $x = 10$, $y = 5$ is the required solution.

Example 3: Find numbers x, y such that

$$7x + 5y = -8 \qquad \text{..........} \ (1)$$
$$5x - 2y = 11 \qquad \text{..........} \ (2)$$

Solution: Suppose we attempt to eliminate y. Take 2 times equation (1) and add 5 times equation (2). This will give $14x + 25x = -16 + 55$, that is, $39x = 39$.

Hence $x = 1$.

Substituting into equation (1) we get $7 + 5y = -8$, that is, $5y = -15$ or $y = -3$. So $x = 1$, $y = -3$ is the required solution.

We could have substituted $x = 1$ into equation (2) instead. In fact we could have eliminated x to get $y = -3$ and then substituted to get $x = 1$.

Example 4: Solve the system

$6x + 9y = 10$ .... (1)

$8x + 12y = 7$ .... (2)

Solution: Multiply (1) by 8.

$48x + 72y = 80$ .... (3)

Multiply (2) by 6. .... (4)

$48x + 72y = 42$ .... (5)

Comparing (3) and (5) we get a contradiction. Hence there is no solution.

This is a case where the two equations represent two distinct parallel lines.

Example 5: Solve the system

$6x + 9y = 3$ .... (1)

$8x + 12y = 4$ .... (2)

Solution: Multiply (1) by 8.

$48x + 72y = 24$ .... (3)

Multiply (2) by 6. .... (4)

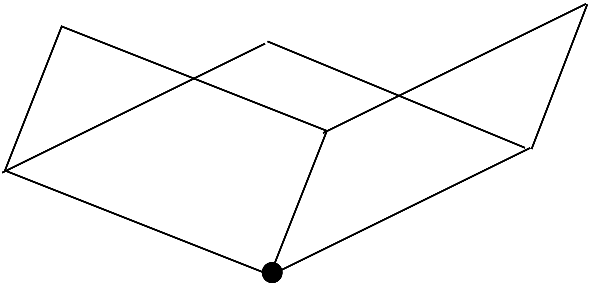
$48x + 72y = 24$ .... (5)

Comparing (3) and (5) we see that the system has reduced to one equation. This corresponds to the situation where the two lines coincide. Hence there are infinitely many solutions.

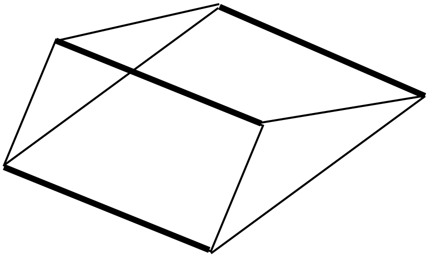
§10.3. Three Linear Equations in Three Variables

The two degenerate cases above, where there are no solutions, or infinitely many, are pretty obvious when we only have two equations in two variables. But when it comes to three equations, in three variables, degenerate cases may not be so obvious.

The equation $ax + by + cz = d$ represents a **plane** in 3-dimensional space. Three such equations represent three planes. The normal situation is for there to be just one point in common, in which the corresponding system has a unique solution.

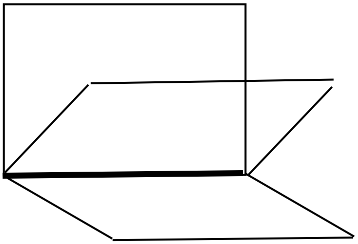


But there's the possibility of no solution, even if no two planes are parallel.



Here the lines of intersection of the pairs of planes are all parallel. A system corresponding to this situation will have no solution. More obvious cases where there is no solution is where two planes (or even all three) are parallel but different.

Then there is the case where the planes are like the pages of a book – all passing through a single line.



A system of equations corresponding to this situation will have infinitely many solutions. Finally there are the more obvious cases where two planes, or all three planes, coincide.

We can see the difficulties that can arise when we have more than two equations and more than two variables. However we'll leave this to a more advanced course to explore.

The technique for solving three linear equations in three variables is the same as before: elimination of variables. However, unless you are very systematic in the way you do it you can get into trouble. What often happens is that you eliminate on variable, and in the course of eliminating a second variable the one you eliminated pops back again. There's a systematic method for eliminating variables, called Gaussian Elimination, which you can read about in my *Linear Algebra: Matrices* notes.

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